

A GEOMETRIC PROOF OF WILBRINK'S CHARACTERIZATION OF EVEN ORDER CLASSICAL UNITALS

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ABSTRACT. Using geometric methods and without invoking deep results from group theory, we prove that a classical unital of even order $n \geq 4$ is characterized by two conditions (I) and (II): (I) is the absence of O'Nan configurations of four distinct lines intersecting in exactly six distinct points; (II) is a notion of parallelism. This was previously proven by Wilbrink (1983), where the proof depends on the classification of finite groups with a split BN-pair of rank 1.

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1. INTRODUCTION

A unital of order $n > 2$ is a design with parameters $2-(n^3 + 1, n + 1, 1)$ (see [12, 4]). If π is a projective plane of order m , i.e. a $2-(m^2 + m + 1, m + 1, 1)$ design, and if a unital U is an induced substructure of π , then we call U an *embedded unital*. Some embedded unitals U of order n are the incidence structure formed from the absolute points and non-absolute lines of a unitary polarity in a projective plane π of order n^2 . Any unital which is isomorphic to such a unital U as a design is called a *polar unital*. Further if the ambient plane is $\text{PG}(2, n^2)$, then the unital is called *classical*. The set of absolute points of a unitary polarity in $\text{PG}(2, n^2)$ is called the *Hermitian curve* (see [15, 17]).

In 1972 [21], O'Nan showed that the classical unital does not contain a configuration of four lines meeting in six points (an *O'Nan configuration*). Piper (1981) [23] conjectured that this property characterizes the classical unital. Wilbrink (1983) [25] characterized the classical unital by three conditions (I), (II) and (III). His proof depends on a result in the classification of finite groups with a split BN-pair of rank 1. Wilbrink [25] further proved that when the order of unital is even, (III) is a necessary condition of (I) and (II).

Let \mathcal{U} be a unital of even order $n \geq 4$ satisfying Wilbrink's conditions (I) and (II). In this article, we give an alternative proof that \mathcal{U} is classical without invoking deep results from group theory, as follows. We construct from \mathcal{U} a hyperbolic Buekenhout unital \mathcal{U}' [9] in $\text{PG}(2, n^2)$, via the Bruck-Bose construction of projective plane [6, 7]. Then we prove that \mathcal{U} is isomorphic to \mathcal{U}' , and hence is classical by a result of Barwick [3]. To construct \mathcal{U}' , we shall consider some inversive planes and a generalized quadrangle derived from \mathcal{U} (Wilbrink [25]), and the special spreads of \mathcal{U} (Hui and Wong [18]). We also need a theorem of Cameron and Knarr [10] on

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how to build a regular spread of $\text{PG}(3, q)$ from a tube in $\text{PG}(3, q)$, and a theorem of Hui [19] on when two inversive planes are identical.

In Section 2, we follow Wilbrink's [25] construction of the inversive planes $\mathcal{I}(x)$ at each point x of \mathcal{U} . Then following the work in [18], we construct a special spread \mathcal{S}_L for each line L of \mathcal{U} using these inversive planes. As a consequence, \mathcal{U} can be embedded in a projective plane π as a polar unital [18]. This enables us to define self-polar triangles with respect to \mathcal{U} intrinsically in terms of Wilbrink's x -parallelism (Theorem 2.2).

In Section 3, by studying the inversive planes $\mathcal{I}(x)$ for various x 's, we prove that the set of points of \mathcal{U} is partitioned into a self-polar triangle, and $n-2$ subsets of $(n+1)^2$ points triply ruled by lines through the vertices of the triangle (Theorem 3.6). This describes how unital lines in π through distinct non-unital points intersect.

In Section 4, we fix one line L of \mathcal{U} and consider the generalized quadrangle $GQ(L)$ as in Wilbrink [25]. Through $GQ(L)$, we associate \mathcal{U} with $Q(4, n)$ formed by the set of points and lines of a parabolic quadric \mathcal{P} in $\text{PG}(4, n)$ [22]. Wilbrink's construction gives naturally a 3-dimensional subspace Σ of $\text{PG}(4, n)$. We find a spread \mathcal{S} in Σ by studying the special spread \mathcal{S}_L . We then prove that \mathcal{S} is regular (Theorem 4.4) using a result on tubes in $\text{PG}(3, n)$ (Cameron and Knarr [10]) and properties of self-polar triangles with respect to \mathcal{U} .

In Section 5, we prove that the partition of \mathcal{U} into a self-polar triangle and triply ruled sets corresponds to a pencil of quadrics in Σ of two lines and $n-1$ hyperbolic quadrics (Theorem 5.5). In particular, this gives a correspondence between the structure of \mathcal{U} and that of Σ . The regularity of \mathcal{S} is essential for proving Theorem 5.5, because we have to describe the reguli of \mathcal{S} in terms by geometry of \mathcal{U} by applying a result of Hui [18] to the Miquelian inversive plane formed by the lines and reguli of \mathcal{S} (Bruck [5]).

In Section 6, by considering the spread \mathcal{S} of Σ in $\text{PG}(4, n)$, we construct a projective plane $\pi(\mathcal{S})$ by the Bruck-Bose construction [6]. Since \mathcal{S} is regular, $\pi(\mathcal{S})$ is $\text{PG}(2, n^2)$ [7]. By Buekenhout [9], \mathcal{P} defines a hyperbolic Buekenhout unital \mathcal{U}' in $\pi(\mathcal{S})$ [9] (also known as a nonsingular Buekenhout unital). By Barwick [3], since $\pi(\mathcal{S}) \cong \text{PG}(2, n^2)$, \mathcal{U}' is the classical unital. With the help of Theorem 5.5, we write down an isomorphism between \mathcal{U} and the classical unital \mathcal{U}' (Theorem 6.2).

2. SELF-POLAR TRIANGLES AND PARALLELISM

Let \mathcal{U} be a unital of even order $n \geq 4$, satisfying Wilbrink's first two conditions (I) and (II) [25]:

- (I) \mathcal{U} contains no O'Nan configurations.
- (II) Let x be a point, L be a line through x , and M be a line missing x , such that L and M meets. For any point $y' \in L \setminus \{x\}$, there is a line M' through y' but not x meeting all lines from x which meet M .

Following Wilbrink [25], we introduce x -parallelism in \mathcal{U} [25]:

Let x be a point, and M, M' be two lines missing x . M, M' are said to be x -parallel if M, M' intersect the same lines through x . We write $M \parallel_x M'$.

\parallel_x defines an equivalence relation on the set of all lines missing x [25]. We denote the equivalence class of a line M under \parallel_x by \overline{M}^x , or simply \overline{M} if there is no confusion.

Further following Wilbrink [25], we introduce an inversive plane $\mathcal{I}(x)$ of order n for every point x in \mathcal{U} ([25], Lemmas 1, 2 and Corollary 3; see also [18]). The points of $\mathcal{I}(x)$ are the lines of \mathcal{U} through x together with a symbol ∞_x . The circle set of $\mathcal{I}(x)$ is $\mathcal{C}^x \cup \mathcal{C}_x$. Here $\mathcal{C}^x, \mathcal{C}_x$ are given by: \mathcal{C}^x is the set of \parallel_x -equivalence classes on the set of lines of \mathcal{U} missing x ; \mathcal{C}_x consists of blocks of the form $C_x(L, L') \cup \{\infty_x\}$, where for any lines L, L' on x , $C_x(L, L') = \{L, L'\} \cup \{L'' \mid L'' \text{ is a line through } x \text{ such that no line of } \mathcal{U} \text{ through } x \text{ meets } L, L' \text{ and } L''\}$. The incidence in $\mathcal{I}(x)$ is defined as follows: Whenever L is a line of \mathcal{U} through x and M is a line of \mathcal{U} missing x , L is incident with \overline{M} in $\mathcal{I}(x)$ if and only if L meets M in \mathcal{U} ; whenever L, L', L'' are lines of \mathcal{U} through x , L is incident with $C_x(L', L'')$ in $\mathcal{I}(x)$ if and only if $L \in C_x(L', L'')$; the point ∞_x is incident with all circles in \mathcal{C}^x but none in \mathcal{C}_x .

According to Dembowski [11], $\mathcal{I}(x)$ is egglike. By Thas [24], every flock in $\mathcal{I}(x)$ is uniquely determined by its carriers. We denote the unique flock in $\mathcal{I}(x)$ with carriers p_1 and p_2 by $\mathcal{F}(p_1, p_2)$.

With the help of flocks of the form $\mathcal{F}(L, \infty_x)$, \mathcal{U} can be shown to satisfy condition (P) [18, Theorem 1.6], formulated in terms of *special spreads* in \mathcal{U} .

Let \mathcal{S} be a spread of \mathcal{U} , with $L \in \mathcal{S}$. Then \mathcal{S} is *special with respect to L* if the following condition is satisfied:

for any point x on L , $\mathcal{S} \setminus \{L\}$ can be partitioned into $n - 1$ subsets $\mathcal{L}_1^x, \dots, \mathcal{L}_{n-1}^x$, each of cardinality n , and the set of lines on x , except L , can be partitioned into $n - 1$ subsets $\mathcal{K}_x^1, \dots, \mathcal{K}_x^{n-1}$, each of cardinality $n + 1$, such that whenever $L' \in \mathcal{L}_i^x$ and $K \in \mathcal{K}_x^j$, L' and K intersect if and only if $i = j$.

Now, [18, Theorem 1.6] says that \mathcal{U} satisfies condition (P), which is a strengthened version of condition (p):

- (p) Let $L_1, L_2, \dots, L_{n^4-n^3+n^2}$ be the lines of \mathcal{U} . There exists a family of lines $\mathcal{F} = \{\mathcal{S}_{L_1}, \mathcal{S}_{L_2}, \dots, \mathcal{S}_{L_{n^4-n^3+n^2}}\}$ such that:
 - (i) For $i = 1, 2, \dots, n^4 - n^3 + n^2$, \mathcal{S}_{L_i} is a spread containing L_i .
 - (ii) For $i \neq j$, $L_i \in \mathcal{S}_{L_j} \setminus \{L_j\}$ if and only if $L_j \in \mathcal{S}_{L_i} \setminus \{L_i\}$.
 - (iii) For any two lines L_i and L_j missing each other, there exists a line L_k such that $L_k \in \mathcal{S}_{L_i} \setminus \{L_i\}$ and $L_k \in \mathcal{S}_{L_j} \setminus \{L_j\}$.
- (P) (p) holds such that for $i = 1, 2, \dots, n^4 - n^3 + n^2$, \mathcal{S}_{L_i} is a special spread with respect to L_i .

In the above statement, the set \mathcal{S}_L is given explicitly by this construction:

Pick some point x on L . \mathcal{S}_L is given by $\overline{L_1} \cup \overline{L_2} \cup \dots \cup \overline{L_{n-1}} \cup \{L\}$, where $\overline{L_1}, \overline{L_2}, \dots, \overline{L_{n-1}}$ are the circles of $\mathcal{F}(L, \infty_x)$ in $\mathcal{I}(x)$.

\mathcal{S}_L is shown to be independent of x [18, Lemma 5.4].

For each line L of \mathcal{U} , denote by \mathcal{S}_L^* the set $\mathcal{S}_L \setminus \{L\}$.

By [18, Theorem 1.1], since \mathcal{U} satisfies (p), \mathcal{U} can be embedded in a projective plane π as a polar unital, so that in π , for each unital line J , the unital lines of π through the pole of J are exactly the lines in \mathcal{S}_J^* . Since two distinct points in π determine a unique line, $\mathcal{S}_J^* \cap \mathcal{S}_{J'}^*$ contains at most one (unital) line for any distinct unital lines J, J' . Thus there are at most three lines in $\mathcal{S}_J \cap \mathcal{S}_{J'}$. Whenever L, M, N are three distinct lines of \mathcal{U} satisfying $\{L, M, N\} = \mathcal{S}_L \cap \mathcal{S}_M = \mathcal{S}_L \cap \mathcal{S}_N = \mathcal{S}_M \cap \mathcal{S}_N$, we say that L, M, N form a *self-polar triangle with respect to \mathcal{U}* .

Lemma 2.1. *Let L, M be disjoint lines of \mathcal{U} . If $M \in \mathcal{S}_L^*$ (or equivalently $L \in \mathcal{S}_M^*$), then there exists a unique line N of \mathcal{U} such that L, M, N form a self-polar triangle with respect to \mathcal{U} .*

Proof. In π , L and M meet in a unique non-unital point a . Let N be the polar line of a . By construction of π , $L, M \in \mathcal{S}_N^*$ and $N \neq L, M$. Since \mathcal{U} satisfies (p), $N \in \mathcal{S}_L \cap \mathcal{S}_M$ by (p)(ii). By (p)(i), $L \in \mathcal{S}_L$, $M \in \mathcal{S}_M$ and $N \in \mathcal{S}_N$. The result follows from the fact there are at most three lines in $\mathcal{S}_L \cap \mathcal{S}_M$, $\mathcal{S}_L \cap \mathcal{S}_N$ and $\mathcal{S}_M \cap \mathcal{S}_N$. \square

Theorem 2.2. *Let L, M, N be disjoint lines of \mathcal{U} . Then the following are equivalent.*

- (1) L, M, N form a self-polar triangle with respect to \mathcal{U} .
- (2) $L \parallel_z M$ for any point $z \in N$.
- (3) Any line meeting two of L, M, N meet all of L, M, N .
- (4) $M \in \mathcal{S}_L^*$, and there exist distinct points $z_1, z_2 \in N$ such that $L \parallel_{z_1} M$ and $L \parallel_{z_2} M$.

Remark 2.3. Since lines in a self-polar triangle play the same role, in Theorem 2.2, statement (1) is also equivalent to (2)': $M \parallel_z N$ for any point $z \in L$, or other statement obtained by permuting L, M, N in (2) and (4).

Proof of Theorem 2.2. Let x be a point on L , and K_1, K_2, \dots, K_{n+1} be the lines through x meeting M .

(1) \Rightarrow (2): Since $M \in \mathcal{S}_L^*$, there is a point $y_i \in K_i$ such that $L \parallel_{y_i} M$ for $i = 1, 2, \dots, n+1$, by Lemmas 5.3 and 5.4 of [18]. Hence, for each i , $\overline{L} = \overline{M}$ in $\mathcal{I}(y_i)$. Then there is a unique point N_i such that $\overline{L} \in \mathcal{F}(N_i, \infty_{y_i})$. This N_i is a line of \mathcal{U} through y_i such that $L \in \mathcal{S}_{N_i}^*$ and $M \in \mathcal{S}_{N_i}^*$. By condition (p), $N_i \in \mathcal{S}_L^* \cap \mathcal{S}_M^*$. Since there is at least one line in $\mathcal{S}_L^* \cap \mathcal{S}_M^*$, we have $N_1 = N_2 = \dots = N_{n+1} = N$.

(2) \Rightarrow (3): For each point $z \in N$, since $L \parallel_z M$, there are $n+1$ lines through z meeting L and M . Thus there are $(n+1)^2$ lines meeting L, M, N . They are the lines meeting at least two of L, M, N .

(3) \Rightarrow (2) follows from the definition of z -parallelism.

(3) \Rightarrow (4): It suffices to show $M \in \mathcal{S}_L^*$. Let $z' \in N$. By (3), any line through z' meeting M meets L . Thus, $L \parallel_{z'} M$. By (3), K_i meets N for $i = 1, 2, \dots, n+1$. Thus z' is the point on K_i such that $L \parallel_{z'} M$. By Lemmas 5.3 and 5.4 of [18], $M \in \mathcal{S}_L^*$.

(4) \Rightarrow (1): Let N' be a line such that L, M, N' form a self-polar triangle. Suppose $z_1 \notin N'$. Let $x_1, x_2 \in L$ be distinct points. For $i = 1, 2$, let J_i be the line passing through x_i and z_1 . Then J_i meets M by (4). Since (1) implies (3), J_1 meets N' at a point, say w . Since (1) implies (2), we have $L \parallel_w M$. Thus, the four lines $J_1, J_2, M, w.x_2$ form an O'Nan configuration, which is a contradiction. ($w.x_2$ denotes the line through w and x_2 .) Hence $z_1 \in N'$. Similarly, $z_2 \in N'$. So $N = N'$. \square

With Theorem 2.2, we characterize \mathcal{S}_M^* by z -parallelism.

Lemma 2.4. *Let L, M be distinct lines of \mathcal{U} . Suppose $M \in \mathcal{S}_L^*$. Then*

$$\mathcal{S}_M^* = \{J \mid J \text{ is a line of } \mathcal{U} \text{ such that there is a point } y \in M \text{ such that } L \parallel_y J\}.$$

Proof. Let J be a line. Suppose there is a point $y \in M$ such that $L \parallel_y J$. Since $L \in \mathcal{S}_M^*$ by condition (p) and $L \parallel_y J$, we have $\overline{J} = \overline{L} \in \mathcal{F}(M, \infty_y)$ in $\mathcal{I}(y)$. Thus, $J \in \mathcal{S}_M^*$ by the construction of \mathcal{S}_M^* .

To prove the reverse inclusion, it suffices to show that there are $|\mathcal{S}_M^*| = n^2 - n$ J 's such that $L \parallel_z J$ for some $z \in M$. Let N be the line such that L, M, N form a self-polar triangle. For each point z on M , there are exactly n lines z -parallel to L . By Theorem 2.2, L and N are two of these n lines. Apart from L and N , no line is z -parallel to L for distinct z 's on M by Theorem 2.2. Hence, there are $2 + (n+1)(n-2) = n^2 - n$ lines z -parallel to L for some z on M , as desired. \square

3. PARTITION \mathcal{U} INTO A SELF-POLAR TRIANGLE AND TRIPLY RULED SETS

In this section, we prove that \mathcal{U} can be partitioned into a self-polar triangle and $n-2$ triply ruled sets of $(n+1)^2$ points (Theorem 3.6). This result will be used at the end of Section 5, where we relate the partition to a pencil of quadrics in a projective space.

In Dover [14, Theorem 3.2] (see also Section 4 of Baker et al. [2]), it is proved that any classical unital admits such a partition by considering coordinates and its automorphism group. The argument in this section will yield a synthetic proof for Dover's result once we prove that \mathcal{U} is classical in Section 6.

We describe how lines of \mathcal{S}_L correspond to circles tangent to \overline{L} in $\mathcal{I}(y)$, whenever y is not a point on L .

Lemma 3.1. *Let L be a line of \mathcal{U} , and y be a point not on L . Let $M \in \mathcal{S}_L$ be the line of \mathcal{U} through y . Let N be the line of \mathcal{U} such that L, M, N form a self-polar triangle. Then the following statements hold:*

- (1) *For every $J \in \mathcal{S}_L \setminus \{L, M, N\}$, \overline{J} is tangent to \overline{L} in $\mathcal{I}(y)$.*
- (2) *Every circle of type \mathcal{C}^y tangent to \overline{L} in $\mathcal{I}(y)$ is $\overline{J'}$ for a unique line J' in $\mathcal{S}_L \setminus \{L, M, N\}$.*

Proof. (1) Let $J \in \mathcal{S}_L \setminus \{L, M, N\}$. It suffices to show that there is a unique line of \mathcal{U} through y meeting both J and L . By Lemma 2.4, since $L \in \mathcal{S}_M^*$ and $J \in \mathcal{S}_L^*$, we have $M \parallel_x J$ for some point $x \in L$. Let K be the line of \mathcal{U} through x and y . Then K meets both J and L .

Suppose there is a line $K' \neq K$ through y meeting both J and L . Let x' be the intersection of K' and L . By Theorem 2.2, K' meets N , and K meets N . By Theorem 2.2, $M \parallel_{x'} N$. Since $M \parallel_x N$ and $M \parallel_x J$, we have $J \parallel_{x'} N$ and so the line $x.(J \cap K')$ meets N . ($J \cap K'$ denotes the intersection point of J and K' .) Then the four lines $K, K', N, x.(J \cap K')$ form an O'Nan configuration. This contradicts (I).

(2) $|\mathcal{S}_L \setminus \{L, M, N\}| = (n^2 - n + 1) - 3 = (n+1)(n-2)$. This is also the number of circles of type \mathcal{C}^y tangent to \overline{L} in $\mathcal{I}(y)$. Since we have (1), to show (2), it suffices to show $\overline{J_1} \neq \overline{J_2}$ for any distinct $J_1, J_2 \in \mathcal{S}_L \setminus \{L, M, N\}$. Suppose not. Then $J_1 \parallel_y J_2$. By (1), there is a line K'' through y meeting J_1, J_2, L . Let x'' be the intersection of K'' and L . Since $J_1, J_2 \in \mathcal{S}_L^*$ and \mathcal{S}_L is a special spread, we have $J_1 \parallel_{x''} J_2$ by the definition of special spread. Let K''' be the line through y and a point of J_2 not on K . Since $J_1 \parallel_y J_2$ and $J_1 \parallel_{x''} J_2$, the four lines $K'', K''', J_1, x''.(J_3 \cap K''')$ form an O'Nan configuration. A contradiction. \square

Using Lemma 3.1 and the characterization of flocks in terms of bundles in even order inversive planes by Dembowski and Hughes [13] (see also (6.2.11), (6.2.12)

with footnote on p.267, and (6.2.13), of [12]), we deduce Lemma 3.2. Lemma 3.2 describes when lines of \mathcal{S}_L^* meet two intersecting lines which do not belong to \mathcal{S}_L^* and which miss L .

Lemma 3.2. *Let L be a line of \mathcal{U} . Let M, N be distinct lines not in \mathcal{S}_L . Suppose M and N intersect at a point y of \mathcal{U} . If \bar{L} belongs to $\mathcal{F}(M, N)$ in $\mathcal{I}(y)$, then the following statements hold:*

- (1) *Let L_1 be the line in \mathcal{S}_L^* through y . In $\mathcal{I}(y)$, L_1 is a point on the circle C determined by M , N and ∞_y .*
- (2) *In \mathcal{U} , every line in \mathcal{S}_L^* meeting M meets N as well.*
- (3) *In \mathcal{U} , there is a unique point x on L such that any line from x meeting L_1 misses all other lines in \mathcal{S}_L^* that meet both M and N . Furthermore, in $\mathcal{I}(y)$, the unital line $x.y$ through x and y is a point on C .*

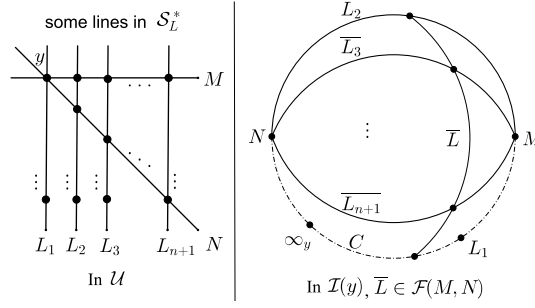


FIGURE 1. illustration of Lemma 3.2

Proof. By Dembowski and Hughes [13], in $\mathcal{I}(y)$, the flock $\mathcal{F}(M, N)$ is the set of circles tangent to every circle in the bundle $\mathcal{B}(M, N)$, i.e. the set of circles in $\mathcal{I}(y)$ through the points M and N . Suppose \bar{L} is in $\mathcal{F}(M, N)$ in $\mathcal{I}(y)$. Then \bar{L} is tangent to every circle in $\mathcal{B}(M, N)$. By Lemma 3.1, $\mathcal{B}(M, N) = \{\bar{L}_2, \bar{L}_3, \dots, \bar{L}_{n+1}, C\}$ for some distinct lines $L_2, L_3, \dots, L_{n+1} \in \mathcal{S}_L^*$. Since \mathcal{S}_L^* is a spread, L_1 does not meet L_2, L_3, \dots, L_{n+1} . Hence in $\mathcal{I}(y)$, L_1 is not on $\bar{L}_2, \bar{L}_3, \dots, \bar{L}_{n+1}$, but on the remaining circle C of $\mathcal{B}(M, N)$. This proves (1). The lines in \mathcal{S}_L^* meeting M are L_1, L_2, \dots, L_{n+1} , and they meet N . This proves (2).

Let K be the unital line on y such that K is the intersection point of C and \bar{L} in $\mathcal{I}(y)$. Since K is on \bar{L} in $\mathcal{I}(y)$, K meets L at a point, say x , in \mathcal{U} . Since K is not on $\bar{L}_2, \bar{L}_3, \dots, \bar{L}_{n+1}$ in $\mathcal{I}(y)$, K does not meet L_2, L_3, \dots, L_{n+1} in \mathcal{U} . Since \mathcal{S}_L^* is a special spread and K is a line from x meeting $L_1 \in \mathcal{S}_L^*$ but not $L_2, L_3, \dots, L_{n+1} \in \mathcal{S}_L^*$, x is a point on L satisfying the property in (3). Uniqueness of x follows from that fact that in $\mathcal{I}(y)$, K is the unique point on \bar{L} not on $\bar{L}_2, \bar{L}_3, \dots, \bar{L}_{n+1}$. This proves (3). \square

We introduce the notion of *triply ruled set* for a general unital: in a unital of order m , a set of $(m+1)^2$ points is *triply ruled* if there are three partitions of the $(m+1)^2$ points by lines. The following lemma suggests a method to find a triply ruled set.

Lemma 3.3. *Let L and M_1 be disjoint lines of \mathcal{U} with $M_1 \notin \mathcal{S}_L$. Let L_1, L_2, \dots, L_{n+1} be the lines of \mathcal{S}_L^* meeting M_1 . Then there are lines $M_2, M_3, \dots, M_{n+1}, N_1, N_2, \dots, N_{n+1}$ such that $\{M_1, M_2, \dots, M_{n+1}\}$ and $\{N_1, N_2, \dots, N_{n+1}\}$ are partitions of the set of points covered by the disjoint lines L_1, L_2, \dots, L_{n+1} .*

Proof. Let y be a point on M_1 . Since M_1 and L are disjoint, the point M_1 is not incident with \bar{L} in $\mathcal{I}(y)$. Since $M_1 \notin \mathcal{S}_L^*$, condition (p) implies $L \notin \mathcal{S}_{M_1}^*$ and hence $\bar{L} \notin \mathcal{F}(M_1, \infty_y)$. Thus, there is a unital line N_1 through y such that $\bar{L} \in \mathcal{F}(M_1, N_1)$. By Lemma 3.2, L_1, L_2, \dots, L_{n+1} meet N_1 . For $i = 2, 3, \dots, n+1$, let y_i be the intersection point of N_1 and L_i . By a similar argument, for each $i = 2, 3, \dots, n+1$, there is a unital line M_i through y_i such that $\bar{L} \in \mathcal{F}(M_i, N_1)$, and there is a unital line N_i through y_i such that $\bar{L} \in \mathcal{F}(M_i, N_i)$. By Lemma 3.2, L_1, L_2, \dots, L_{n+1} meet M_i, N_i , for all $i = 2, 3, \dots, n+1$.

We are going to show that M_1, M_2, \dots, M_{n+1} are mutually disjoint. Suppose M_{i_1} and M_{i_2} intersect at a point z for some distinct $i_1, i_2 \in \{1, 2, \dots, n+1\}$. Then z is on L_{i_3} for some $i_3 \in \{1, 2, \dots, n+1\} \setminus \{i_1, i_2\}$. Let $i_4 \in \{1, 2, \dots, n+1\} \setminus \{i_1, i_2, i_3\}$. Then the four lines $M_{i_1}, M_{i_2}, N_1, L_{i_4}$ form an O'Nan configuration. A contradiction. Hence $\{M_1, M_2, \dots, M_{n+1}\}$ is a partition of the set of points covered by L_1, L_2, \dots, L_{n+1} . By a similar argument, the same conclusion can be drawn for N_i 's. \square

Note that there cannot be a forth partition of the set of points covered by L_1, L_2, \dots, L_{n+1} ; otherwise, there would be an O'Nan configuration constituted by lines of different partitions. This suggests the following notion of parallelism:

Let M, N be lines missing L and not in \mathcal{S}_L . We say that M and N are L -parallel, denoted by $M \parallel_L N$, if they are identical or they are non-intersecting and meet the same lines in \mathcal{S}_L^* . We say that M and N are L -non-parallel if they intersect and meet the same lines in \mathcal{S}_L^* .

Lemma 3.4. *Let L be a line of \mathcal{U} . Then \parallel_L defines an equivalence relation in the set of lines missing L and not in \mathcal{S}_L . Each class has $n+1$ lines. There are $n(n-1)(n-2)$ equivalence classes under the equivalence relation \parallel_L .*

Proof. It is clear that \parallel_L is reflexive and symmetric. It is transitive because of Lemma 3.3 and the fact that two points determine a line. Since non-equal L -parallel line are non-intersecting, there are at most $n+1$ lines in an equivalence class. By Lemma 3.3, this upper bound is achieved. The number of lines missing L is $n(n-1)(n^2-n-1)$. Among these lines, $n(n-1)$ are in \mathcal{S}_L^* . The result follows from simple counting. \square

By Lemma 3.2, for any distinct lines M' and N' through y , N' is L -non-parallel to M' if $\bar{L} \in \mathcal{F}(M', N')$ in $\mathcal{I}(y)$. Furthermore, the set of lines which are L -non-parallel to M' is an L -parallel class. In the context of Lemma 3.3, it is natural to ask whether M_1, M_2, \dots, M_{n+1} are in \mathcal{S}_M^* for some line M . The answer is yes:

Lemma 3.5. *Refer to the set-up in Lemma 3.3. Let M be the line in \mathcal{S}_L^* such that $M_1 \in \mathcal{S}_M^*$. Then for each $i = 2, 3, \dots, n+1$, M_i belongs to \mathcal{S}_M^* .*

Proof. By Lemma 2.4, there is a point $z_1 \in M$ such that $L \parallel_{z_1} M_1$. Let z_2, z_3, \dots, z_{n+1} be the points on M other than z_1 . Let $M_{z_1} = M_1$. Let N be a line such that L, M, N form a self-polar triangle. For $i = 2, 3, \dots, n+1$, we claim that there is a line $M_{z_i} \in \mathcal{S}_M^*$ distinct from L and N , such that M_{z_i} is z_i -parallel to L , and M_{z_i}

meets L_1, L_2, \dots, L_{n+1} . If the claim is true, then $\{M_{z_i} | z_i \in M\}$ is the L -parallel class containing M_{z_1} . Indeed, if $M_{z_i} = M_{z_j}$ for some $i \neq j$, then M_{z_i}, L and N would be three lines both z_i -parallel and z_j -parallel, giving an O'Nan configuration. Hence $M_{z_i} \neq M_{z_j}$ for distinct i, j . Furthermore, for $i = 2, 3, \dots, n+1$, since M_{z_i} is z_i -parallel to L , M_{z_i} is in \mathcal{S}_M^* by Lemma 2.4. Thus M_{z_i} 's are non-intersecting. Hence $\{M_{z_i} | z_i \in M\}$ is the L -parallel class containing M_{z_1} . By uniqueness of such a class, $\{M_1, M_2, \dots, M_{n+1}\} = \{M_{z_i} | z_i \in M\}$. The result follows.

We now prove the claim. Let $k \in \{2, 3, \dots, n+1\}$. For $i = 1, 2, \dots, n+1$, let $K_i = (M_1 \cap L_i).z_1$. Since $L \parallel_{z_1} M_1$, we can label the points on L as x_1, x_2, \dots, x_{n+1} such that $x_i \in K_i$. Since \mathcal{S}_L is a special spread, and K_i passes through $x_i \in L$ and meets both $M \in \mathcal{S}_L^*$ and $L_i \in \mathcal{S}_L^*$, we have $M \parallel_{x_i} L_i$. For $i = 1, 2, \dots, n+1$, let $K'_i = z_k.x_i$. Since $M \parallel_{x_i} L_i$ and K'_i meets M , we conclude that K'_i meets L_i , say at w_i . If w_1, w_2, \dots, w_{n+1} are collinear, then take M_{z_k} to be the line that they are on and we find M_{z_k} (Figure 3).

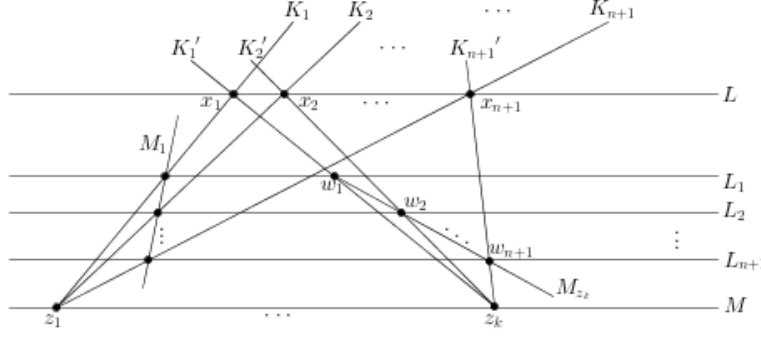


FIGURE 2. M_{z_k} is L -parallel to M_1

It remains the case when some of w_1, w_2, \dots, w_{n+1} are not collinear. For $i = 1, 2, \dots, n+1$, let J_i be the line on w_i that is z_k -parallel to L . At least two of J_1, J_2, \dots, J_{n+1} are the same line. For, otherwise, there would be $n+1$ lines z_k -parallel to L . Without loss of generality, assume $J_1 = J_2$. We are going to prove that J_1 meets L_1, L_2, \dots, L_{n+1} by contradiction. Suppose J_1 misses some of L_3, L_4, \dots, L_{n+1} . By Lemma 3.3, there are lines M_2 and N_2 on w_1 respectively L -parallel to M_1 and L -non-parallel to M_1 . Thus, in $\mathcal{I}(w_1)$, $\mathcal{B}(M_2, N_2) = \{\overline{L_2}, \overline{L_3}, \dots, \overline{L_{n+1}}, C\}$ for some circle C of type \mathcal{C}_{w_1} . Since L_1 are disjoint from L_2, L_3, \dots, L_{n+1} in \mathcal{U} , L_1 is a point on C in $\mathcal{I}(w_1)$. Since $J_1 = J_2$ meets L_2 in \mathcal{U} , J_1 is a point on $\overline{L_2}$ in $\mathcal{I}(w_1)$. Since $J_1 \neq M_2$ and $J_1 \neq N_2$ by hypothesis, and $J_1 \in \overline{L_2}$, we conclude $J_1 \notin C$. On the other hand, since $J_1 \parallel_{z_k} L$, $\mathcal{B}(K'_1, J_1) = \{\overline{K'_2}, \overline{K'_3}, \dots, \overline{K'_{n+1}}, D\}$ for some circle D of type \mathcal{C}_{w_1} . Note that $L_1 \in D$. Indeed, if L_1 is not on D , then L_1 meets K'_i in \mathcal{U} for some $i \neq 1$. By Theorem 2.2, K'_i meets N . Since K'_i meets both $N \in \mathcal{S}_L^*$ and $L_1 \in \mathcal{S}_L^*$, we have $N \parallel_{x_i} L_1$ because \mathcal{S}_L is a special spread. Thus the four lines $K'_1, K'_i, L_1, x_i.(K_1 \cap N)$ form an O'Nan configuration. Hence $L_1 \in D$. Since $J_1 \in D$ but $J_1 \notin C$, we have $C \neq D$. Since ∞_{w_1} and L_1 are points on both C and D , $K'_1 \notin C$. Then K'_1 meets L_j in \mathcal{U} for some $j \neq 1$. By Theorem 2.2, K'_i meets N . Since K'_1 meets both $N \in \mathcal{S}_L^*$ and $L_j \in \mathcal{S}_L^*$, we have $N \parallel_{x_1} L_j$ and so the four lines $K'_1, K'_j, L_j, x_1.(K'_j \cap N)$ form an

O'Nan configuration. A contradiction. Thus J_1 meets L_1, L_2, \dots, L_{n+1} . Take M_{z_k} to be J_1 and we prove our claim. \square

Theorem 3.6. *Let L, M, N be a self-polar triangle with respect to \mathcal{U} . Then $\mathcal{S}_L \setminus \{L, M, N\}$, $\mathcal{S}_M \setminus \{L, M, N\}$, $\mathcal{S}_N \setminus \{L, M, N\}$ can be respectively partitioned into $n - 2$ subsets*

$$\begin{aligned} & \{L_1^1, L_2^1, \dots, L_{n+1}^1\}, \dots, \{L_1^{n-2}, L_2^{n-2}, \dots, L_{n+1}^{n-2}\}; \\ & \{M_1^1, M_2^1, \dots, M_{n+1}^1\}, \dots, \{M_1^{n-2}, M_2^{n-2}, \dots, M_{n+1}^{n-2}\}; \\ & \{N_1^1, N_2^1, \dots, N_{n+1}^1\}, \dots, \{N_1^{n-2}, N_2^{n-2}, \dots, N_{n+1}^{n-2}\}; \end{aligned}$$

each of cardinality $n + 1$, such that for $i = 1, 2, \dots, n - 2$, the sets of points incident respectively on the lines of $\{L_1^i, L_2^i, \dots, L_{n+1}^i\}$, $\{M_1^i, M_2^i, \dots, M_{n+1}^i\}$ and $\{N_1^i, N_2^i, \dots, N_{n+1}^i\}$ are the same.

Proof. Take a line $M_1^1 \in \mathcal{S}_M \setminus \{L, M, N\}$. Note that $M_1^1 \notin \mathcal{S}_L^*$ because $\mathcal{S}_L^* \cap \mathcal{S}_M^* = \{N\}$. Let $L_1^1, L_2^1, \dots, L_{n+1}^1$ be the lines of \mathcal{S}_L^* meeting M_1^1 . By Lemmas 3.3 and 3.5, there is an L -parallel class $\{M_1^1, M_2^1, \dots, M_{n+1}^1\} \subset \mathcal{S}_M^*$ and its L -non-parallel class $\{N_1^1, N_2^1, \dots, N_{n+1}^1\}$ both partitioning the set of points covered by $L_1^1, L_2^1, \dots, L_{n+1}^1$. Since $\{N_1^1, N_2^1, \dots, N_{n+1}^1\}$ is an L -parallel class and an M -parallel class, it is a subset of $\mathcal{S}_{N'}^*$ where $N' \in \mathcal{S}_L^* \cap \mathcal{S}_M^*$ by applying Lemma 3.5 twice. Hence $N' = N$. Repeat the process $n - 3$ times by taking a line $M_l^i \in \mathcal{S}_M \setminus (\{L, M, N\} \cup \{M_k^l \mid k = 1, 2, \dots, i - 1, l = 1, 2, \dots, n + 1\})$. This finishes the proof. \square

Remark 3.7. We may interpret Theorem 3.6 as follows. If \mathcal{U} is embedded in a projective plane π as a polar unital via the construction of [18, Theorem 1.1], then \mathcal{S}_L^* , \mathcal{S}_M^* , \mathcal{S}_N^* are respectively the set of lines through the pole of L , M , and N . Thus, Theorem 3.6 says that in π , the set of points of \mathcal{U} is partitioned into a self-polar triangle, and $n - 2$ subsets of $(n + 1)^2$ points triply ruled by lines through the vertices of the triangle.

Remark 3.8. In the setting in Theorem 3.6, for any disjoint index sets I_L, I_M, I_N such that $I_L \cup I_M \cup I_N = \{1, 2, \dots, n + 1\}$, the set $\{L, M, N\} \cup \{L_j^i \mid i \in I_L, j = 1, 2, \dots, n + 1\} \cup \{M_j^i \mid i \in I_M, j = 1, 2, \dots, n + 1\} \cup \{N_j^i \mid i \in I_N, j = 1, 2, \dots, n + 1\}$ is a spread of \mathcal{U} . These spreads are the subregular spreads studied by Dover [14].

4. FROM A SPECIAL SPREAD OF A UNITAL TO A REGULAR SPREAD OF $\text{PG}(3, n)$

From now on, we fix a line L , and let x_1, x_2, \dots, x_{n+1} be the points on L .

Following Wilbrink [25], we are going to construct a generalized quadrangle $GQ(L)$, which is isomorphic to $Q(4, n)$ [25]. We will then embed $Q(4, n)$ into $\text{PG}(4, n)$ and choose a 3-dimensional projective space Σ in $\text{PG}(4, n)$. It turns out that the special spread \mathcal{S}_L of \mathcal{U} introduced in Section 2 defines a regular spread \mathcal{S} of Σ (Theorem 4.4). Using the Bruck-Bose construction [6, 7], we will construct a projective plane in Section 6 using this regular spread \mathcal{S} , such that \mathcal{U} is embedded in a way into this projective plane as a classical unital.

To construct $GQ(L)$, we recall the definition of the sets \mathcal{A}_{ij} , $1 \leq i, j \leq n + 1$ of \mathcal{U} ([25], also see [18]). Considering $\mathcal{I}(x_1)$, denote the circles in the bundle $\mathcal{B}(L, \infty_{x_1})$ by $\{L, \infty_{x_1}\} \cup \mathcal{A}_{1j}$, where $j = 1, 2, \dots, n + 1$. We have defined $\mathcal{A}_{11}, \mathcal{A}_{12}, \dots, \mathcal{A}_{1, n+1}$. Next, for each $j \in \{1, 2, \dots, n + 1\}$, consider the pencil $\langle L, \{L, \infty_{x_1}\} \cup \mathcal{A}_{1j} \rangle$ in $\mathcal{I}(x_1)$, i.e. the maximal set of mutually tangent circles through L with a member the circle $\{L, \infty_{x_1}\} \cup \mathcal{A}_{1j}$. For $k = 1, 2, \dots, n - 1$, denote by C_{jk} the remaining

circles in the pencil. For $i \in \{2, 3, \dots, n+1\}$, consider the $n-1$ lines on x_i which correspond respectively to these $n-1$ circles C_{jk} 's. Denote this set of lines by \mathcal{A}_{ij} . We have defined $\mathcal{A}_{2j}, \mathcal{A}_{3j}, \dots, \mathcal{A}_{n+1,j}$, for $j = 1, 2, \dots, n+1$. The definition of \mathcal{A}_{ij} is independent on the choice of the point $x_1 \in L$.

The set of points of $GQ(L)$ is

$$\{\mathcal{A}_{ij} \mid i, j = 1, 2 \cdots n+1\} \cup \{y \mid y \text{ is a point of } \mathcal{U} \text{ not on } L\}.$$

The set of lines of $GQ(L)$ is

$$\{A_i \mid i = 1, 2 \cdots n+1\} \cup \{B_i \mid i = 1, 2 \cdots n+1\} \cup \{K \mid K \neq L \text{ is a line of } \mathcal{U} \text{ meeting } L\}.$$

The incidence of $GQ(L)$ is as follows. \mathcal{A}_{ij} is incident with A_k if and only if $i = k$; \mathcal{A}_{ij} is incident with B_k if and only if $j = k$; for any line K of \mathcal{U} meeting L , \mathcal{A}_{ij} is incident with K if and only if $K \in \mathcal{A}_{ij}$; a point y of \mathcal{U} is never incident with A_i or B_j for $i, j = 1, 2, \dots, n+1$; incidence between a point and a line of \mathcal{U} is the natural incidence.

Consider a parabolic quadric \mathcal{P} in $\text{PG}(4, n)$. The points and lines of \mathcal{P} form a generalized quadrangle $Q(4, n)$ [22]. By [25], $GQ(L)$ is isomorphic under some GQ isomorphism

$$(4.1) \quad \varphi : GQ(L) \longrightarrow Q(4, n)$$

to $Q(4, n)$.

Consider the 3-dimensional subspace Σ of $\text{PG}(4, n)$ determined by the skew lines $\varphi(A_1)$ and $\varphi(A_2)$. Then $\Sigma \cap \mathcal{P} = \{\varphi(\mathcal{A}_{ij}) \mid i, j = 1, 2 \cdots n+1\}$, and is a hyperbolic quadric \mathcal{H} with regulus

$$(4.2) \quad \mathcal{R}_0 = \{\varphi(A_i) \mid i = 1, 2, \dots, n+1\}$$

and opposite regulus $\{\varphi(B_i) \mid i = 1, 2, \dots, n+1\}$. \mathcal{H} defines a polarity

$$(4.3) \quad \alpha : \Sigma \longrightarrow \Sigma$$

of Σ .

The tangent spaces of \mathcal{P} are concurrent at a point *nucleus* \mathbf{N} of \mathcal{P} . Let

$$(4.4) \quad \mu : \mathcal{P} \longrightarrow \Sigma$$

be the function defined as follows: for any point \mathbf{V} of \mathcal{P} , $\mu(\mathbf{V})$ is the intersection point of Σ and the line joining \mathbf{N} and \mathbf{V} . Since $\mathcal{H} \cap \Sigma = \mathcal{H}$, μ is identity on \mathcal{H} . Since $|\mathcal{P}| = n^3 + n^2 + n + 1 = |\Sigma|$, μ is a bijection. Hence, we have a 1-1 correspondence between points of $GQ(L)$ and that of Σ via the composition function $\mu\varphi$. Furthermore, some quadratic cones in \mathcal{P} are mapped to planes of Σ because n is even:

Lemma 4.1. *Let $\mathbf{V} \in \mathcal{P} \setminus \Sigma$. Let \mathcal{Q} be the quadratic cone formed by the intersection of \mathcal{P} and the tangent space of \mathcal{P} at \mathbf{V} . Then μ maps \mathcal{Q} onto the plane $\alpha(\mu(\mathbf{V}))$ in Σ . Furthermore, the plane $\alpha(\mu(\mathbf{V}))$ meets \mathcal{H} in an irreducible conic with nucleus $\mu(\mathbf{V})$.*

Proof. Since $\mathbf{V} \notin \Sigma$, every generator of \mathcal{Q} meets Σ in a unique point (of \mathcal{H}). Hence, for any generator l of \mathcal{Q} , $\mu(l)$ is tangent to \mathcal{H} and passes through $\mu(\mathbf{V})$. By a property of hyperbolic quadric of even order, $\{\mu(l) \mid l \text{ is a generator of } \mathcal{Q}\}$ is on the plane $\alpha(\mu(\mathbf{V}))$. Since $|\mathcal{Q}| = n^2 + n + 1$, the image set $\mu(\mathcal{Q})$ is a plane. Furthermore, $\mu(\mathbf{V})$ is the nucleus of the irreducible conic formed by the intersection of $\mu(\mathcal{Q})$ and \mathcal{H} . \square

Using Lemma 4.1 and the GQ isomorphism φ , we prove that every line of \mathcal{S}_L^* is mapped to a line of Σ under $\mu\varphi$:

Lemma 4.2. *Let M be a line of \mathcal{U} in \mathcal{S}_L^* . Then $\mu(\varphi(M))$ is a line in Σ .*

Proof. By Theorem 2.2, $L \parallel_{z_i} M$ for $i = 1, 2, \dots, n+1$. Let \mathcal{Q}_i be the quadratic cone formed by the intersection of \mathcal{P} and the tangent space of \mathcal{P} at $\varphi(z_i)$. Hence $\varphi(M) \subset \mathcal{Q}_i$. By Lemma 4.1, $\mu(\varphi(M)) \subset \bigcap_{i=1}^{n+1} \alpha(\mu(\varphi(z_i)))$ and $\alpha(\mu(\varphi(z_i)))$'s are planes in Σ . Since $|\varphi(M)| = n+1$ and $\mu(\varphi(M))$ is in the intersection of $n+1$ planes, $\mu(\varphi(M))$ is a line. \square

Since \mathcal{S}_L is a spread of \mathcal{U} and μ is a bijection, the set $\{\mu(\varphi(L')) \mid L' \in \mathcal{S}_L^*\}$ consists of disjoint lines. Let

$$(4.5) \quad \mathcal{S} = \mathcal{R}_0 \cup \{\mu(\varphi(L')) \mid L' \in \mathcal{S}_L^*\}.$$

Then \mathcal{S} is a spread. We claim that \mathcal{S} is regular (Theorem 4.4). The justification of this claim requires the notion of tube [10]:

When q is even, a *tube* in $\text{PG}(3, q)$ is a pair $\mathcal{T} = \{l, \mathcal{B}\}$, where $\{l\} \cup \mathcal{B}$ is a collection of mutually disjoint lines of $\text{PG}(3, q)$ such that for each plane Π of $\text{PG}(3, q)$ containing l , the intersection of Π with the lines of \mathcal{B} is a hyperoval. For any mutually skew lines l_1, l_2, l_3 in Σ , denote by $\mathcal{R}(l_1, l_2, l_3)$ the unique regulus determined by them. According to Cameron and Knarr [10], if $\{l, \{l_0, l_1, \dots, l_{q+1}\}\}$ is a tube, then the union $\bigcup_{i=1}^{n+1} \mathcal{R}(l, l_0, l_i)$ is a regular spread in $\text{PG}(3, q)$.

Lemma 4.3. *Let M, N be lines of \mathcal{U} . Suppose L, M, N form a self-polar triangle. Then the pair $\mathcal{T} = \{\mu(\varphi(M)), \{\mu(\varphi(N))\} \cup \mathcal{R}_0\}$ is a tube in Σ .*

Proof. Let z_1, z_2, \dots, z_{n+1} be the points of N . By Lemma 4.1, the $n+1$ planes in Σ containing $\mu(\varphi(M))$ are $\alpha(\mu(\varphi(z_i)))$, $i = 1, 2, \dots, n+1$. Since \mathcal{R}_0 is a regulus of \mathcal{H} , $\alpha(\mu(\varphi(z_i)))$ meets the points covered by \mathcal{R}_0 in an irreducible conic with nucleus $\mu(\varphi(z_i))$ by Lemma 4.1. The result follows. \square

Theorem 4.4. *\mathcal{S} is a regular spread in Σ .*

Proof. Let M, N be lines such that L, M, N form a self-polar triangle. For $i = 1, 2, \dots, n+1$, let $\mathcal{R}_i = \mathcal{R}(\mu(\varphi(M)), \mu(\varphi(N)), \mu(\varphi(A_i)))$. By [10] mentioned above, the union $\bigcup_{i=1}^{n+1} \mathcal{R}_i$ is a regular spread. We are done if we show $\mathcal{S} = \bigcup_{i=1}^{n+1} \mathcal{R}_i$. Thus it suffices to show $\mathcal{S} \subset \bigcup_{i=1}^{n+1} \mathcal{R}_i$.

Let $L_1 \in \mathcal{S}_L \setminus \{L, M, N\}$. Since $L \in \mathcal{S}_M^*$ and $L_1 \in \mathcal{S}_L^*$, there is a point $x_i \in L$ such that $M \parallel_{x_i} L_1$ by Lemma 2.4. Let K_1, K_2, \dots, K_{n+1} be the lines of \mathcal{U} on x_i meeting M (and hence meeting L_1). Then they meet N by Theorem 2.2. Hence, $\mu(\varphi(K_1)), \mu(\varphi(K_2)), \dots, \mu(\varphi(K_{n+1}))$ are lines in Σ meeting $\mu(\varphi(M))$, $\mu(\varphi(N))$ and $\mu(\varphi(A_i))$. Furthermore, these $n+1$ lines are disjoint. Indeed, the points K_1, K_2, \dots, K_{n+1} are incident with $\overline{M} \in \mathcal{F}(L, \infty_{x_i})$ in $\mathcal{I}(x_i)$, where \overline{M} is tangent to each circle in the bundle $\mathcal{B}(L, \infty_{x_i})$ [13], and so we may assume $K_j \in \mathcal{A}_{ij}$ for $j = 1, 2, \dots, n+1$. Thus, $\{\mu(\varphi(K_j)) \mid j = 1, 2, \dots, n+1\}$ is the opposite regulus of \mathcal{R}_i . Since $\mu(\varphi(L_1))$ meets every line in the opposite regulus of \mathcal{R}_i , it is in \mathcal{R}_i . \square

5. FROM A PARTITION OF \mathcal{S}_L^* TO A PENCIL OF QUADRICS IN $\text{PG}(3, n)$

We use the notations in Section 4 and continue to prove that \mathcal{U} is classical. The key result in this section is Theorem 5.5. It says that the image of the partition of \mathcal{S}_L in Theorem 3.6 under $\mu\varphi$ corresponds to a pencil of quadrics of two lines and

$n - 1$ hyperbolic quadrics of the projective space Σ , where μ , φ and Σ are defined Section 4. To prove Theorem 5.5, we have to describe every regulus of the spread \mathcal{S} defined in (4.5), in terms of the geometry of \mathcal{U} . A regulus of \mathcal{S} has two, one, or no common lines with \mathcal{R}_0 . We consider these cases separately.

We first consider the reguli of \mathcal{S} with exactly one common line with \mathcal{R}_0 . They can be described using x -parallelism introduced in Section 2, where the x 's are the points of the line L .

Lemma 5.1. *Let $i \in \{1, 2, \dots, n+1\}$ and $L_1, L_2, \dots, L_n \in \mathcal{S}_L^*$. If $\mathcal{R} = \{\mu(\varphi(A_i)), \mu(\varphi(L_1)), \mu(\varphi(L_2)), \dots, \mu(\varphi(L_n))\}$ forms a regulus in \mathcal{S} , then $L_j \parallel_{x_i} L_k$ for any $j, k \in \{1, 2, \dots, n\}$.*

Proof. Let l be a line in the opposite regulus of \mathcal{R} . By the definition of μ and the construction of $GQ(L)$, there is a line K of \mathcal{U} on x_i such that $l = \mu(\varphi(K))$. Since l meets $\mu(\varphi(L_1)), \mu(\varphi(L_2)), \dots, \mu(\varphi(L_n))$, the line K meets $L_1, L_2, \dots, L_n \in \mathcal{S}_L^*$ in \mathcal{U} . Since \mathcal{S}_L is a special spread, L_1, L_2, \dots, L_n are x_i -parallel. \square

Lemma 5.3 describes the reguli of \mathcal{S} with exactly two common lines with \mathcal{R}_0 . To prove Lemma 5.3, we need Lemma 5.2.

Lemma 5.2. *Let $i_1, i_2, j_1, j_2 \in \{1, 2, \dots, n+1\}$ with $i_1 \neq i_2$ and $j_1 \neq j_2$. Let $K_1 \in \mathcal{A}_{i_1 j_1}$ and $K_2 \in \mathcal{A}_{i_2 j_2}$ be lines of \mathcal{U} meeting at a point y of \mathcal{U} . Let M be a line in \mathcal{S}_L^* not through y . Then \overline{M} is in the flock $\mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$ if and only if there is a point $z \in M$ such that $z.x_k \in \mathcal{A}_{i_k j_k}$ for $k = 1, 2$.*

Proof. Without loss of generality, assume $i_1 = j_1 = 1$ and $i_2 = j_2 = 2$. Suppose $\overline{M} \in \mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$. For $k = 1, 2$, let N_k be a line through x_k such that $\overline{M} \in \mathcal{F}(K_k, N_k)$ in $\mathcal{I}(x_k)$. Applying Lemma 3.2(2) respectively to $\overline{M} \in \mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$, $\overline{M} \in \mathcal{F}(K_1, N_1)$ in $\mathcal{I}(x_1)$ and $\overline{M} \in \mathcal{F}(K_2, N_2)$ in $\mathcal{I}(x_2)$, any line of \mathcal{S}_M^* meeting K_1 meets K_2, N_1 and N_2 . Since $M \in \mathcal{S}_L^*$, we have $L \in \mathcal{S}_M^*$ by condition (p). Hence, L is the line in \mathcal{S}_M^* that meet K_1, K_2, N_1, N_2 . By Lemma 3.2(3), there is a point z_k on M such that any line from z_k meeting L miss all other lines in \mathcal{S}_M^* that meet K_1, K_2, N_1, N_2 . By uniqueness in Lemma 3.2(3), $z_1 = z_2$. By Lemma 3.2(1) and (3), in $\mathcal{I}(x_k)$, the points $z_k.x_k, K_k, L, \infty_z$ are concircular. By definition of \mathcal{A}_{kk} , we have $z_k.x_k \in \mathcal{A}_{kk}$ for $k = 1, 2$.

Conversely, note that there are exactly $n - 2$ circles in $\mathcal{F}(K_1, K_2)$ in $\mathcal{I}(y)$ is of type \mathcal{C}^y , and each of these circles gives one $z \neq y$ such that $z.x_1 \in \mathcal{A}_{11}$ and $z.x_2 \in \mathcal{A}_{22}$. To prove the converse, it suffices to show there are exactly $n - 2$ such z 's. By definition of \mathcal{A}_{kk} , in $\mathcal{I}(x_1)$, lines of \mathcal{A}_{22} correspond to circles of a pencil with carrier L , and lines of \mathcal{A}_{11} correspond to points on a circle through L not in that pencil. Hence, each line of \mathcal{A}_{11} meets exactly one line of \mathcal{A}_{22} in \mathcal{U} . Since $|\mathcal{A}_{11}| = n - 1$ and there is only one line of \mathcal{A}_{11} passing through y , there are exactly $n - 2$ such z 's. \square

Lemma 5.3. *Let $i_1, i_2 \in \{1, 2, \dots, n+1\}$ with $i_1 \neq i_2$. Let $L_1, L_2, \dots, L_{n-1} \in \mathcal{S}_L^*$. Suppose \mathcal{R} is a regulus of \mathcal{S} containing $\mu(\varphi(A_{i_1})), \mu(\varphi(A_{i_2})), \mu(\varphi(L_1)), \mu(\varphi(L_2)), \dots, \mu(\varphi(L_{n-1}))$. Then for any point $z_1 \in L_1$, if K_1 is the line passing through z_1 and x_{i_1} , and if K_2 is the line passing through z_1 and x_{i_2} , then in $\mathcal{I}(z_1)$,*

$$\mathcal{F}(K_1, K_2) = \{\overline{L_2}, \overline{L_3}, \dots, \overline{L_{n-1}}, C\}$$

for some circle C through ∞_{z_1} .

Proof. Let z be a point on L_1 . Let l be the line in the opposite regulus of \mathcal{R} through $\mu(\varphi(z))$. Let $z' \in L_2$ be the unital point such that $\mu(\varphi(z')) \in l$. Consider α defined in (4.3). Then $\alpha(l)$ meets \mathcal{H} at $\mu(\varphi(\mathcal{A}_{i_1 j_1}))$ and $\mu(\varphi(\mathcal{A}_{i_1 j_2}))$ for some j_1, j_2 with $j_1 \neq j_2$, and $\alpha(l)$ lies on the plane $\alpha(\mu(\varphi(z')))$. Let \mathcal{Q} be the quadratic cone formed by the intersection of \mathcal{P} and the tangent space of \mathcal{P} at $\varphi(z')$. By Lemma 4.1, $\mu^{-1}(\alpha(l)) \in \mathcal{Q}$ and so $\varphi(\mathcal{A}_{i_1 j_1}), \varphi(\mathcal{A}_{i_1 j_2}) \in \mathcal{Q}$. Hence z' lies on some unital lines $K_1 \in \mathcal{A}_{i_1 j_1}$ and $K_2 \in \mathcal{A}_{i_1 j_2}$. By Lemma 5.2, $\overline{L_2} \in \mathcal{F}(K_1, K_2)$ in $\mathcal{I}(z)$. Similarly, $\overline{L_3}, \dots, \overline{L_{n-1}} \in \mathcal{F}(K_1, K_2)$. The result follows by Lemma 3.1. \square

Lemma 5.4 gives a characterization of reguli of \mathcal{S} with no common line with \mathcal{R}_0 , by considering inversive planes whose blocks are defined by the reguli of \mathcal{S} . For each line J of \mathcal{U} that misses L and not in \mathcal{S}_L , let $C(J)$ be the set of images of the $n+1$ lines of \mathcal{S}_L^* meeting J under $\mu\varphi$.

Lemma 5.4. *A set of $n+1$ lines of $\mathcal{S} \setminus \mathcal{R}_0$ is a regulus if and only if it is $C(J)$ for some line J of \mathcal{U} missing L and not in \mathcal{S}_L .*

Proof. Consider the incidence structure

$$(5.1) \quad \mathcal{I}_1 = (\mathcal{S}, \mathcal{C})$$

where \mathcal{C} is the set of the reguli of \mathcal{S} . By Theorem 4.5 (iv) of Bruck [5], \mathcal{I}_1 is the Miquelian inversive plane of order n . Note that \mathcal{R}_0 is a circle of \mathcal{I}_1 . We denote by \mathcal{C}_0 the set of those circles in \mathcal{I}_1 disjoint from \mathcal{R}_0 ; by \mathcal{C}_1 the set of those circles in \mathcal{I}_1 tangent to \mathcal{R}_0 ; by \mathcal{C}_2 the set of those circles in \mathcal{I}_1 secant to \mathcal{R}_0 .

Let $\mathcal{C}_0^* = \{C(J) \mid J \text{ is a line of } \mathcal{U} \text{ missing } L \text{ and not in } \mathcal{S}_L\}$. Now considering the incidence structure

$$(5.2) \quad \mathcal{I}_2 = (\mathcal{S}, (\mathcal{C} \setminus \mathcal{C}_0) \cup \mathcal{C}_0^*).$$

By Theorem 2 of [19], provided that \mathcal{I}_2 is an inversive plane of order n , we will have $\mathcal{I}_1 = \mathcal{I}_2$ and thus $\mathcal{C}_0 = \mathcal{C}_0^*$. Hence, to prove Lemma 5.4, it suffices to prove that \mathcal{I}_2 is a $3-(n^2+1, n+1, 1)$ design.

Since \mathcal{S} has n^2+1 lines, \mathcal{I}_2 has n^2+1 points.

A block in \mathcal{C}_0^* has exactly $n+1$ points because every line which is not in \mathcal{S}_L and which misses L meets exactly $n+1$ lines of \mathcal{S}_L^* . Other blocks of \mathcal{I}_2 has exactly $n+1$ points because every regulus of \mathcal{S} consists of $n+1$ lines.

$|\mathcal{C}_0^*| = n(n-1)(n-2)/2$ because there are $n(n-1)(n-2)$ L -parallel classes by Lemma 3.4, and any class and its L -non-parallel class define a same block of \mathcal{I}_2 . Since \mathcal{I}_1 is an inversive plane, $|\mathcal{C} \setminus \mathcal{C}_0| = n^2(n+3)/2$. Thus \mathcal{I}_2 has $n(n^2+1)$ blocks.

It remains to show that any two distinct blocks of \mathcal{I}_2 have at most two common points. Since \mathcal{I}_2 only differs from the inversive plane \mathcal{I}_1 in blocks missing \mathcal{R}_0 , we only need to consider the case when one of the two blocks belongs to \mathcal{C}_0^* . Let $C(J)$ be a block in \mathcal{C}_0^* . Let $\mu(\varphi(L_k)), k = 1, 2, 3$ be distinct points of $C(J)$, where $L_1, L_2, L_3 \in \mathcal{S}_L^*$.

Suppose $\mu(\varphi(L_k)), k = 1, 2, 3$, are on a block in \mathcal{C}_1 . By Lemma 5.1, L_1, L_2, L_3 are x -parallel for some x on L . Since J meets L_1, L_2, L_3 but does not pass through x , there is an O'Nan configuration, contradicting (I).

Suppose $\mu(\varphi(L_k)), k = 1, 2, 3$, are on a block in \mathcal{C}_2 which contains $\mu(\varphi(A_i))$ and $\mu(\varphi(A_j))$. Let z be a point intersection of J and L_1 . Let K_i, K_j be the lines through z that pass through x_i and x_j respectively. By Lemma 5.3, $\overline{L_2}, \overline{L_3}$ are

distinct circles in $\mathcal{F}(K_i, K_j)$ in $\mathcal{I}(z)$. Since circles in a flock are disjoint, $\overline{L_2}$ and $\overline{L_3}$ are disjoint. This contradicts that J is a line through z meeting L_2 and L_3 .

Suppose $\mu(\varphi(L_k)), k = 1, 2, 3$, are on a block $C(J_2)$ in \mathcal{C}_0^* . Suppose J and J' are not L -parallel or L -non-parallel. We claim that $L_2 \parallel_y L_3$ for any point $y \in L_1$. If the claim is true, then L_1, L_2, L_3 is a self-polar triangle by Theorem 2.2 and so $\mathcal{S}_{L_2}^* \cap \mathcal{S}_{L_3}^* = \{L_1\}$. Since $L \in \mathcal{S}_{L_2}^* \cap \mathcal{S}_{L_3}^*$ by condition (p) and $L \neq L_1$, a contradiction rises. It follows that J and J' are L -parallel or L -non-parallel, and $C(J) = C(J')$.

To see that $L_2 \parallel_y L_3$ for any $y \in L_1$. Note that, if y is a point on L_1 , then the four lines on y which are respectively L -parallel to J_1 , L -non-parallel to J_1 , L -parallel to J_2 and L -non-parallel to J_2 , meet both L_2 and L_3 . Thus by Wilbrink [25, Lemma 1], $L_2 \parallel_y L_3$.

Hence, a block in \mathcal{C}_0^* has at most two common points with any other block. Thus, \mathcal{I}_2 is an inversive plane of order n . \square

To prove the main theorem in this section, we need Lemma 5.4 and that fact that when q is even, if a regular spread of $\text{PG}(3, q)$ is partitioned into two lines and $q - 1$ reguli, then the two lines and the hyperbolic quadrics containing the reguli lie in a pencil of quadrics (Hirschfeld [16, Lemma 17.1.5, Corollary of Theorem 17.1.6]).

Theorem 5.5. *Let M, N be lines in \mathcal{S}_L^* . Suppose L, M, N form a self-polar triangle with respect to \mathcal{U} . Consider the lines $L_1^1, L_2^1, \dots, L_{n+1}^1, \dots, L_1^{n-2}, \dots, L_{n+1}^{n-2}, M_1^1, M_2^1, \dots, M_{n+1}^1$ as obtained in the construction described in Theorem 3.6. For $i = 1, 2, \dots, n-1$, let \mathcal{H}_i be the image set of the points on $L_1^i, L_2^i, \dots, L_{n+1}^i$ under $\mu\varphi$. Consider \mathcal{H} is defined in Section 4. Then $\{\mathcal{H}_i \mid i = 1, 2, \dots, n-2\} \cup \{\mathcal{H}\} \cup \{\mu(\varphi(M)), \mu(\varphi(N))\}$ is a pencil of quadrics in Σ of two lines and $n-1$ hyperbolic quadrics.*

Proof. By Lemma 5.4, for $i = 1, 2, \dots, n-2$, the block $C(M_1^i) = \{\mu(\varphi(L_1^i)), \mu(\varphi(L_2^i)), \dots, \mu(\varphi(L_{n+1}^i))\}$ is a regulus. Thus \mathcal{H}_i is a hyperbolic quadric. By the definition of \mathcal{S} , $\mathcal{S} = \left(\bigcup_{i=1}^{n-2} C(M_1^i)\right) \cup \mathcal{R}_0 \cup \{\mu(\varphi(M)), \mu(\varphi(N))\}$. Since \mathcal{S} is regular by Theorem 4.4, the result follows by [16] mentioned above. \square

6. COMPLETION OF THE PROOF THAT \mathcal{U} IS CLASSICAL

In this section, we use the notations in Section 4 and complete the proof that \mathcal{U} is classical. Recall from Section 1 that we wish to show that \mathcal{U} is isomorphic to the hyperbolic Buekenhout unital \mathcal{U}' in $\text{PG}(2, n^2) \cong \overline{\pi(\mathcal{S})}$ defined by \mathcal{P} in Section 4 under the Bruck-Bose representation [6, 7]. To this end, we define an isomorphism φ' from φ , where φ is the isomorphism between $GQ(L)$ and $Q(4, n)$ defined in (4.4).

Lemma 6.1. *Let J be a line of \mathcal{U} missing L and not in \mathcal{S}_L . Then $\varphi(J)$ lies on a plane which contains a line of \mathcal{S} .*

Proof. We locate $\varphi(J)$ as a subset of an intersection of two quadratic cones, as follows. Let M be the line in \mathcal{S}_L^* such that $J \in \mathcal{S}_M^*$. By Lemma 2.4, there is a point $z \in M$ such that $L \parallel_z J$. Let \mathcal{Q}_0 be the quadratic cone formed by the intersection of \mathcal{P} and the tangent space of \mathcal{P} at $\varphi(z)$. By the definition of $GQ(L)$,

$$(6.1) \quad \varphi(J) \subset \mathcal{Q}_0.$$

By Lemma 4.1, $\mu(\mathcal{Q}_0) \cap \mathcal{H}$ is a base of \mathcal{Q}_0 and its nucleus is $\mu(\varphi(z))$. On the other hand, let \mathcal{Q}_1 be the cone with vertex \mathbf{N} and a base $\mu(\varphi(J))$. By the definition of μ , we have

$$(6.2) \quad \varphi(J) \subset \mathcal{Q}_1.$$

We are going to show \mathcal{Q}_1 is a quadratic cone by studying $\mu(\varphi(J))$. By Lemma 4.1 and (6.1), $\mu(\varphi(J))$ lies on the plane $\mu(\mathcal{Q}_0)$. Let $\mathcal{H}_1 = \{\mu(\varphi(y)) \mid y \text{ is a point of } \mathcal{U} \text{ on a line of } \mathcal{S}_L^* \text{ meeting } J\}$. By Lemma 5.4, \mathcal{H}_1 is a hyperbolic quadric in Σ . Note that $\mu(\varphi(J)) \subset \mathcal{H}_1$. To see that the plane $\mu(\mathcal{Q}_0)$ is secant to \mathcal{H}_1 , consider the line N of \mathcal{U} such that L, M, N form a self-polar triangle with respect to \mathcal{U} . By Theorem 2.2, $L \parallel_z N$. Thus, \mathcal{Q}_0 contains $\varphi(N)$, and the plane $\mu(\mathcal{Q}_0)$ contains $\mu(\varphi(N))$. Since $L \parallel_z J$ and $N \in \mathcal{S}_L^*$, we conclude that N is disjoint from any line of \mathcal{S}_L^* that meets J . Thus, $\mu(\varphi(N))$ is external to \mathcal{H}_1 . Thus, $\mu(\varphi(J)) = \mu(\mathcal{Q}_0) \cap \mathcal{H}_1$ and the base $\mu(\varphi(J))$ of \mathcal{Q}_1 is an irreducible conic (Figure 6).

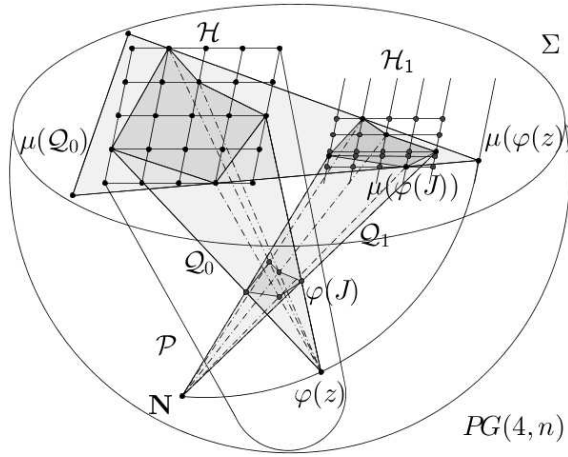


FIGURE 3. $\varphi(J) \subset \mathcal{Q}_0 \cap \mathcal{Q}_1$

We then study $\mathcal{Q}_0 \cap \mathcal{Q}_1$. Let Π be the plane determined by three distinct points in $\mathcal{Q}_0 \cap \mathcal{Q}_1$. Then $\mathcal{Q}_0 \cap \Pi$ and $\mathcal{Q}_1 \cap \Pi$ are irreducible conics. Since L is z -parallel to J , the nucleus of $\mu(\varphi(J))$ is $\mu(\varphi(z))$. Since the nuclei of $\mathcal{Q}_0 \cap \mu(\mathcal{Q}_0)$ and $\mathcal{Q}_1 \cap \mu(\mathcal{Q}_0)$ are both $\mu(\varphi(z))$, the conics $\mathcal{Q}_0 \cap \Pi$ and $\mathcal{Q}_1 \cap \Pi$ have a same nucleus, namely the intersection of Π and the line through \mathbf{N} , $\varphi(z)$ and $\mu(\varphi(z))$. Since $\mathcal{Q}_0 \cap \Pi$ and $\mathcal{Q}_1 \cap \Pi$ are irreducible conics with the same nucleus and containing three distinct common points, $\mathcal{Q}_0 \cap \Pi = \mathcal{Q}_1 \cap \Pi$ by Lemma 2.1 of Luyckx [20]. Thus, $\varphi(J) \subset \Pi$.

By Theorem 5.5, $\mathcal{H}_1, \mathcal{H}, \mu(\varphi(M)), \mu(\varphi(N))$ are quadrics in a same pencil. Thus, \mathcal{H}_1 and \mathcal{H} meet $\mu(\varphi(N))$ in the same conjugate pair of points with respect to \mathbb{F}_{q^2} [8, Theorem 5.1]. Hence the bases of \mathcal{Q}_0 and \mathcal{Q}_1 pass through the conjugate pair of points in $PG(4, q^2)$, and so do $\mathcal{Q}_0 \cap \mathcal{Q}_1$ and Π . Hence, the plane Π , where $\varphi(J)$ lies on, contains the line $\mu(\varphi(N)) \in \mathcal{S}$. \square

We prove that a unital \mathcal{U} is classical by studying the hyperbolic Buekenhout unital \mathcal{U}' in $\pi(\mathcal{S})$ defined by \mathcal{P} , where $\pi(\mathcal{S})$ is the projective plane constructed by the Bruck-Bose construction [6, 7] (see André [1] for an alternative treatment).

Theorem 6.2. \mathcal{U} is classical.

Proof. Consider the incidence structure $\pi(\mathcal{S})$ whose points are the affine points of $\text{PG}(4, q)$ and whose lines are the affine planes of $\text{PG}(4, q)$ each containing a line of \mathcal{S} . The incidence of $\pi(\mathcal{S})$ is the incidence of $\text{PG}(4, q)$. By Bruck and Bose [6], $\pi(\mathcal{S})$ is an affine translation plane of order q^2 . Complete $\pi(\mathcal{S})$ to a projective plane $\overline{\pi(\mathcal{S})}$ by adding a line at infinity, L_∞ . Since \mathcal{S} is regular, $\overline{\pi(\mathcal{S})}$ is Desarguesian [7]. By Buekenhout [9], \mathcal{P} corresponds to a hyperbolic Buekenhout unital \mathcal{U}' in $\overline{\pi(\mathcal{S})}$. Let a_i be the points on L_∞ corresponding to the $\varphi(A_i)$'s in \mathcal{R}_0 . Then the point set of \mathcal{U}' is $(\mathcal{P} \setminus \Sigma) \cup \{a_1, a_2, \dots, a_{q+1}\}$. Since \mathcal{S} is regular by Theorem 4.4, \mathcal{U}' is classical by Barwick [3].

Let $\varphi' : \mathcal{U} \rightarrow \mathcal{U}'$ be a function defined by $\varphi'(x_i) = a_i$ for $i = 1, 2, \dots, n+1$; $\varphi'(y) = \varphi(y)$ for any point y of \mathcal{U} not on L ; $\varphi'(J) = \{\varphi(y) \mid y \in J\}$ for any line J of \mathcal{U} missing L ; $\varphi'(K) = \{\varphi(y) \mid y \in K \setminus L\} \cup \{a_i\}$ for any line K of \mathcal{U} meeting L at some point x_i ; $\varphi'(L) = \{a_1, a_2, \dots, a_{q+1}\}$.

Note that φ' is a well-defined function. Indeed, for any line $J_1 \notin \mathcal{S}_L^*$ of \mathcal{U} missing L , its image $\varphi'(J_1)$ is on a secant plane on a line of \mathcal{S} by Lemma 6.1, and hence is a line of \mathcal{U}' . As for a line $J_2 \in \mathcal{S}_L^*$ of \mathcal{U} missing L , its image $\varphi'(J_2)$ is on the secant plane of \mathcal{P} determined by the point \mathbf{N} and the line $\mu(\varphi(J_2)) \in \mathcal{S}$, and hence $\varphi'(J_2)$ is a line of \mathcal{U}' . As for a line K meeting L at some x_i , its image $\varphi'(K)$ consists of a_i and the n affine points of $\text{PG}(4, n)$ on a line of \mathcal{P} meeting $\varphi(A_i)$, and hence is a line of \mathcal{U}' . Since φ is an isomorphism, φ' preserves incidence. Clearly, φ' is injective. Thus, φ' is a design isomorphism and \mathcal{U} is classical. \square

Remark 6.3. By [18, Theorem 1.1], since \mathcal{U} satisfies (p) , \mathcal{U} can be embedded in a projective plane π as a polar unital. The author does not know whether π is Desarguesian or not.

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